INSTITUTE OF PHYSICS PUBLISHING EUROPEAN JOURNAL OF PHYSICS

Eur. J. Phys. **24** (2003) 115–123 PII: S0143-0807(03)52946-4

Cylinder on an incline as a fold catastrophe system

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Received 29 August 2002, in final form 4 October 2002
Published 13 January 2003
Online at stacks.iop.org/EJP/24/115

Abstract

The motion of a cylinder on an inclined plane, acted upon by a torque along its axis, is studied theoretically and experimentally. It is shown that the potential for the centre-of-mass exhibits the features of a fold catastrophe potential, the control parameter being related to the strength of the torque. This parameter determines whether or not the system experiences stable equilibrium positions. If it does, and depending on the initial conditions, it may perform oscillations around an equilibrium position, or it may cross a no-return point and roll down. A cylinder with a magnet inside, placed on an inclined plane in a region where a uniform magnetic field is present, is a real example of such a system. We constructed that system and report the data obtained in a set of experiments.

1. Introduction

Potentials of the generic form $U(x) = x^3 + bx$, where x is the dynamical variable for a particle and b is the 'control parameter', are well known in catastrophe theory [1–3]: $U(x)$ is the so-called 'fold catastrophe' potential. There are not many physical systems, either in classical or modern physics, showing the fold catastrophe behaviour. An interesting example, recently reported, is the Cartesian diver [4]. In this work, we present another example: a cylinder on an incline with a torque along its axis.

From the generic form of the fold catastrophe potential it is clear that the potential has a local maximum and a local minimum if b is smaller than the critical value $b^* = 0$. When the control parameter becomes larger than b^* there is a transition and the potential has no extremes. Irrespective of the value of the control parameter, $x = 0$ is always an inflexion point. For $b < 0$, the stable equilibrium point is $x_e = \sqrt{|b|/3}$ (minimum of U) and the unstable one is $x'_e = -\sqrt{|b|/3}$ (maximum of U). In the region $-\infty < x < x'_e$ there are no equilibrium positions and x'_e is a 'no-return point'. The particle remains in the region $x > x'_e$

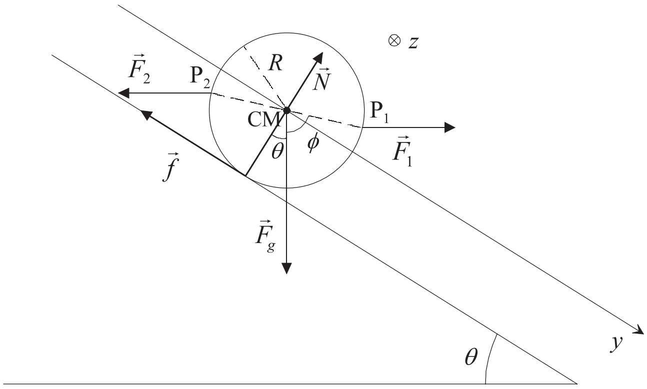

Figure 1. Forces acting upon the cylinder.

for suitable initial conditions and, if there is a slight friction, it goes eventually to x_e (with no friction, the system oscillates around the equilibrium position). Keeping the control parameter fixed, the system will never return to the region $x > x'_e$ when it reaches the region $x < x'_e$ with zero or negative velocity. At the critical value of the control parameter, the maximum and minimum merge together at the inflexion point $x = 0$, which becomes a stationary point. For $b > 0$, $U(x)$ is an increasing function of x and there are no equilibrium points.

We will show that a cylinder placed on an inclined plane, additionally acted upon by a torque along its axis, may oscillate around an equilibrium position, or may roll all the way down. The shape of the centre-of-mass (CM) effective potential crucially depends on the system parameters: by changing them, a local minimum of the potential may disappear, causing a sudden change in the cylinder kinematics. That behaviour may be interpreted in the framework of catastrophe theory.

We have constructed such a system consisting of a cylinder, with a magnet inside, placed on an inclined plane in a region where there exists an uniform magnetic field produced by Helmholtz coils. With that set-up the fold catastrophe behaviour was experimentally verified.

The paper is organized as follows. In section 2 we describe the mechanical system with the fold catastrophe behaviour and, in section 3, we study its kinematics in more detail, integrating numerically the equations of motion. In section 4 we describe the experimental set-up used to perform the experiments and show the data. The conclusions are presented in section 5.

2. The system and its dynamics

Let us take a cylinder of radius R and mass m (the moment of inertia with respect to its axis is $I = \frac{1}{2}mR^2$), which may roll without slipping on a plane inclined at an angle θ to the horizontal. The weight of the cylinder is $F_g = mg$. A torque, which can be regarded as resulting from two *constant* horizontal forces $\vec{F}_1$ and $\vec{F}_2$, equal in magnitude ($F_1 = F_2 = F$), also acts upon the cylinder. These forces are always applied at the *same* points on the cylinder: at P_1, the force points to the right and, at P_2, the force points to the left. However, the pertinent point for the following discussion is the existence of a torque along the cylinder axis like the one produced by forces $\vec{F}_1$ and $\vec{F}_2$. The forces acting upon the cylinder are shown in figure 1.

We denote by ϕ the angle between a reference direction, e.g. the vertical direction pointing downwards and the direction of the line joining the CM and P_1. In figure 1, besides the weight, $\vec{F}_g$, and the forces $\vec{F}_1$ and $\vec{F}_2$, the normal reaction, $\vec{N}$, and the static frictional force, $\vec{f}$, are also shown. The force $\vec{N}$ is perpendicular to the inclined plane and the frictional force is tentatively represented pointing upwards but only the specific kinematical conditions may allow us to determine the actual direction of that force [5]. We assume that the static friction coefficient, μ_s, is always large enough so that the condition $f \leqslant \mu_s mg\cos\theta$ is satisfied, i.e. the cylinder rolls without slipping.

We define the direction y as indicated in the figure and, consistently, the angular acceleration should be considered positive if the cylinder rotates clockwise around its axis, i.e. if the angular acceleration points to the z direction (see figure 1). For the translation of the CM along y, Newton's Second Law yields

$$-f + mg\sin\theta = ma, \tag{1}$$

where a is the (linear) acceleration. The torque due to forces $\vec{F}_1$ and $\vec{F}_2$ is

$$T = 2FR\cos\phi = T_0\cos\phi. \tag{2}$$

We stress that the relevant quantity in the formalism is the torque and not the forces themselves. The physical origin of the torque is at present not important, but in section 4 we describe how the torque given by equation (2) can be applied to a cylinder in a real experiment. For the rotation around the cylinder axis, one has

$$fR - T_0\cos\phi = I\alpha, \tag{3}$$

where α is the angular acceleration. Because rolling takes place without slipping, $a = R\alpha$. Using this expression in equation (3) and inserting the resulting expression in (1), one finds

$$ma = \frac{2}{3}mg\sin\theta - \frac{2T_0}{3R}\cos\phi. \tag{4}$$

Hence, the CM moves as a *particle* of mass m acted upon by the following force along the y axis:

$$F_R(y) = \frac{2}{3}mg\sin\theta - \frac{2T_0}{3R}\cos\phi(y). \tag{5}$$

Note that ϕ is a function of y. If there is no slipping, one has $\phi = \frac{y}{R} + \phi_0$, where ϕ_0 specifies the orientation of the plane containing P_1 and the cylinder axis with respect to the vertical direction for $y = 0$.

The potential energy associated with force (5) is

$$U(y) = -\frac{2}{3}mg\sin\theta y + \frac{2}{3}T_0\sin\left(\frac{y}{R} + \phi_0\right). \tag{6}$$

It is more convenient to introduce the dimensionless quantities $x = (y/R) + \phi_0$, $A = T_0/mgR\sin\theta$ and $V = U/\frac{2}{3}mgR\sin\theta$. Thus, after dropping constant terms, the dimensionless 'potential' is given by

$$V(x) = -x + A\sin x. \tag{7}$$

This potential is sometimes referred to as the 'tilted washboard potential' [6]. The extremes of the potential are the solutions of $\mathrm{d}V(x)/\mathrm{d}x = 0$, and one finds $x = \arccos(1/A)$. For $A < 1$ there are no solutions, hence the potential has no extremes: it is a monotonic decreasing function of x. On the other hand, for $A > 1$, the potential shows up minima and maxima at

$$x_n = -\varphi \pm 2n\pi\,(\text{minima}) \qquad \text{and} \qquad x_{n'} = \varphi \pm 2n'\pi\,(\text{maxima}), \tag{8}$$

where $0 \leqslant \varphi \leqslant \frac{\pi}{2}$ and $n, n' = 0, 1, 2, \ldots$.

The dimensionless potential $V(x)$ is represented in figure 2 for three values of A. The relation with the fold catastrophe potential $U(x) = x^3 + bx$ mentioned in the introduction is obvious. Now, the parameter A plays the role of control parameter, its critical value being $A^* = 1$. For $A > A^*$ (broken curve in figure 2) the extremes of the potential are the points given by equations (8). For $A < A^*$ (dotted curve in figure 2) the potential is a decreasing function of x with no equilibrium positions. For $A = A^*$ (full curve in figure 2), $\varphi = 0$ and $x = 0, \pm 2\pi, \pm 4\pi, \ldots$ are stationary and inflexion points.

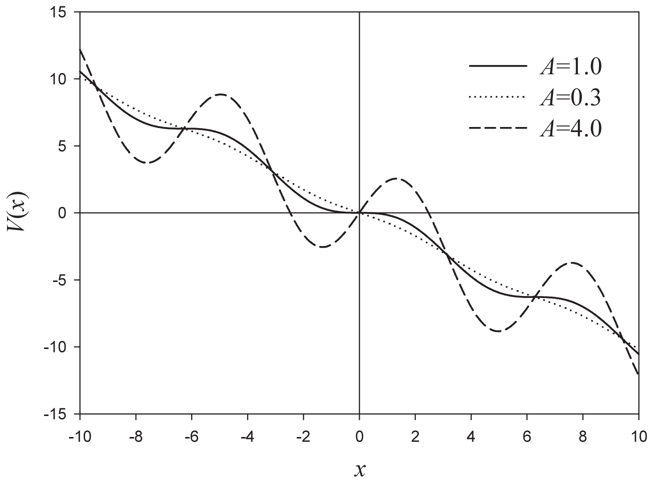

Figure 2. Dimensionless potential, $V(x)$, for different values of the control parameter A.

For $A < A^*$ the system can never stay in static equilibrium. For $A > A^*$ it may remain in static equilibrium or perform oscillations around a potential minimum. However, if it has enough energy to transpose the first potential energy barrier on the right of that local minimum, the cylinder is bound to roll down without return.

So far, the discussion has been kept on general grounds in the sense that no mention has been made of the physical origin of the torque applied to the cylinder. Later on, in section 4, we describe a real system showing up the fold catastrophe behaviour.

3. Kinematics

In order to find the dependence of the position of the CM with time, $y = y(t)$, one has to solve Newton's equation (see equation (4)). In terms of variable x introduced in section 2 this equation is

$$\frac{d^2x}{dt^2} = C(1 - A\cos x), \tag{9}$$

where $C = 2g\sin\theta/3R$ and $A = T_0/mgR\sin\theta$ is the control parameter already introduced in section 2. The parameter C, which essentially depends on the geometry, is just a global factor in Newton's equation and does not determine whether or not the potential has got stability points. For a given cylinder and incline, the control parameter A can be varied by changing T_0.

We performed a numerical integration of equation (9) using the Euler–Cromer algorithm [7], $v_{n+1} = v_n + a_n\Delta t$ and $x_{n+1} = x_n + v_{n+1}\Delta t$. For a potential with stable minima ($A = 4$ and $C = 1$ s^{-2}), typical results for $x = x(t)$ are shown in figure 3 for different initial 'velocities' (note that $v = dx/dt$ is expressed in s^{-1} but, for the sake of simplicity, we call it the velocity).

As an initial condition for x, we consider that the system is always at the stable equilibrium position $x_0 = x_e = -1.318$ (see figure 2 and equation (8)). For an initial velocity $v_0 = -1.00$ s^{-1} it performs oscillations around the equilibrium point x_e. The amplitude of the oscillations increases if the magnitude of the velocity increases. For $v_0 = -3.18$ s^{-1} the system still remains in the attraction basin of the potential, having a periodic motion (eventually it would stop, at x_e, if a kinetic frictional force were present). However, for initial velocities bigger than $v_0 = -3.19$ s^{-1}, the energy is enough to transpose the potential barrier on the right

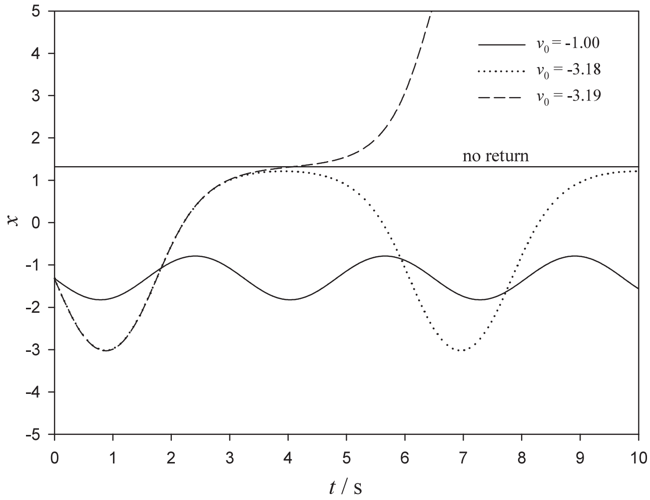

Figure 3. The dimensionless position x as a function of time, for $A = 4$ and $C = 1\ \mathrm{s}^{-2}$, and various initial velocities. The time step $\Delta t = 0.1$ s was used in the Euler–Cromer algorithm.

of x_e. The maximum of the potential barrier, located at $x'_e = -x_e = 1.318$ (see figure 2 and equation (8)), is a no-return point. As long as the system goes beyond this point, it will never return to the region where it was initially: the cylinder rolls all the way down, as indicated by the broken curve in figure 3.

The frequency of the small oscillations around a stable equilibrium position is given by $\omega = \sqrt{C\sqrt{A^2 - 1}}$. For $A = 4$ and $C = 1\ \mathrm{s}^{-2}$ one finds $\tau = 2\pi/\omega = 3.19$ s, which is the period of the almost harmonic oscillation represented by the full curve in figure 3.

4. Experiment

We constructed a system behaving as described in the previous sections. It is a cylinder having a magnet inside with its magnetic moment, $\vec{\mu}$, perpendicular to the cylinder axis. This cylinder is placed on an incline, in a region filled with a constant and stationary magnetic field, as is represented in figure 4.

The torque resulting from the interaction between the magnetic moment and the magnetic field is along the cylinder axis and it is given by $T = |\vec{\mu} \times \vec{B}| = \mu B \cos\phi$, where ϕ is the angle shown in figure 4 (see also figure 1). From the comparison of this expression with equation (2) one concludes that $T_0 = \mu B$ and, therefore, the control parameter becomes $A = \mu B / mgR \sin\theta$.

Figure 5 shows the experimental set-up used by us, consisting of the cylinder with the magnet inside, the inclined plane and the Helmholtz coils, which produce an approximately uniform magnetic field in the region between them.

For the Helmholtz coils used in our apparatus, the magnetic field is related to the current, I, in the coils by $B = \kappa I$, where $\kappa = (7.8 \pm 0.2) \times 10^{-4}\ \mathrm{T\,A}^{-1}$, and therefore the critical parameter is given by

$$A = \frac{\mu \kappa I}{mgR\sin\theta}. \tag{10}$$

The value of κ can be obtained theoretically from the diameter of the coils (30 cm, in our case) and the number of turns in each coil (130, in our case). We checked the uniformity of the magnetic field in the region between the coils and also the value of the constant κ, which

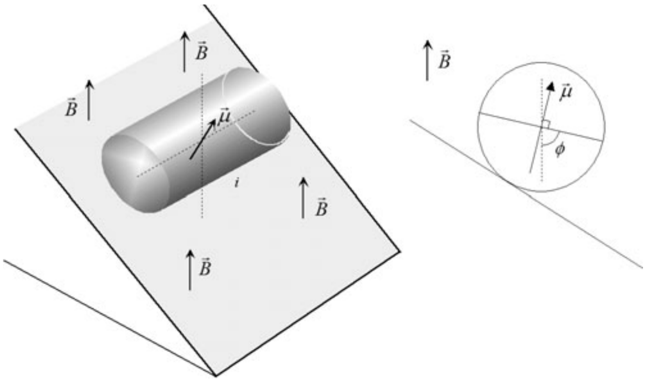

Figure 4. Cylinder with a magnet inside whose magnetic moment, $\vec{\mu}$, is perpendicular to the axis. A vertical uniform stationary magnetic field, $\vec{B}$, fills the region of the incline. The angle between the magnetic field and the magnetic moment is $\frac{\pi}{2} - \phi$ (see also figure 1).

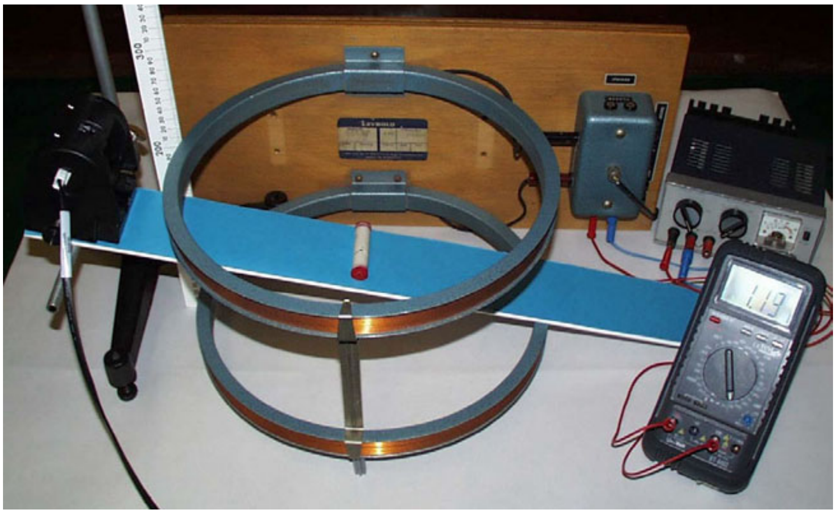

Figure 5. Experimental set-up showing the cylinder on the incline and the pair of Helmholtz coils. The current in the coils, provided by the power supply, is measured with the ammeter (both on the right side of the picture). The apparatus on the left, mounted on top of the incline, is a sonar sensor used to measure the position and velocity of the cylinder.

(This figure is in colour only in the electronic version)

proved to yield accurate values for the magnetic field within 3%. The current I in the coils was directly measured with an ammeter.

In our experiments we used a cylinder made of rigid plastic tube of radius $R = 7.50 \pm 0.05$ mm and ~6 cm long. A small, disc-shaped, $Nd_2Fe_{14}B$ magnet (6 ± 0.05 mm radius, 5 ± 0.05 mm height), was carefully placed in the correct position inside the tube which was then uniformly filled with plasticine. The mass of the 'filled' cylinder is $m = 18.37 \pm 0.01$ g.

In an independent experiment we determined the magnetic dipole moment of the neodymium magnet, μ, that enters in equation (10), by measuring its magnetization, M, following the procedure recently described by Connors in [8]. In brief, the B field along the disc axis was accurately measured as a function of the distance z from the centre of the magnet

Table 1. For various angles (in deg) of the inclined plane, the minimal currents required to keep the cylinder in static equilibrium are listed as well as the corresponding values for the control parameter as given by equation (10). The results in the last column should be compared with the theoretical value $A = 1$.

θ	I/A	A
7.5 ± 0.3	0.36 ± 0.01	0.90 ± 0.07
9.1 ± 0.3	0.45 ± 0.01	0.94 ± 0.07
10.0 ± 0.3	0.49 ± 0.01	0.93 ± 0.07
11.4 ± 0.3	0.57 ± 0.01	0.95 ± 0.07
13.2 ± 0.3	0.64 ± 0.01	0.92 ± 0.07
14.4 ± 0.4	0.70 ± 0.01	0.93 ± 0.06
15.1 ± 0.4	0.76 ± 0.01	0.96 ± 0.06
16.7 ± 0.4	0.83 ± 0.01	0.95 ± 0.06

with a GaAs Hall probe connected to a PHYWE 1610.93 teslameter. The magnetization of the disc was obtained from a fit of the data to the theoretical expression of $B(z)$ for a disc-shaped homogeneous magnet. As explained in [8], there is a single fitting parameter, namely the remnant induction$B_{\mathrm{r}} = \mu_0 M$. We have obtained a good fit to our data for $B_{\mathrm{r}} = 1.25 \pm 0.05$ T, leading to the magnetization $M = (9.9 \pm 0.4) \times 10^5$ A m^{-1}. From this value and from the volume, V, of the magnet we obtained for the magnetic dipole moment $\mu = VM = 0.57 \pm 0.03$ A m^2. Our values for B_{r} and μ compare rather well with typical data for $Nd_2Fe_{14}B$ magnets [9] and are also close to those determined by Connors [8] for a similar magnet.

For various angles θ of the inclined plane we measured the minimal current, I, required to keep the cylinder still in static equilibrium but about to roll down, i.e. the current that still keeps the cylinder in the limit of static equilibrium. Table 1 shows, for several angles, these currents and the calculated values for A, using experimental data in equation (10). Theoretically, $A = 1$, corresponding to the critical point, and our results are in agreement with that.

For $A > 1$ there are equilibrium positions for the cylinder, as explained in the previous sections. Thus, it may perform oscillations around one of such equilibrium points with frequency (see the end of section 3)

$$f = \frac{1}{2\pi}\sqrt{\frac{2g\sin\theta}{3R}\sqrt{\left(\frac{\mu\kappa I}{mgR\sin\theta}\right)^2 - 1}} \tag{11}$$

if the oscillation amplitudes are small.

In figure 6(a) the velocity as a function of time is represented for a real experiment with the cylinder oscillating around one of the potential minima. In this particular case, experimental conditions were $A = 2.7 \pm 0.2$. The cylinder was initially slightly displaced from its equilibrium position and released from rest. The position and velocity of the cylinder were measured with a sonar sensor (see figure 5) connected to a PC through a PASCO750 interface. The frequency of the (damped) oscillations, obtained directly from the data shown in figure 6(a), is 2.9 ± 0.1 Hz, comparing well with 2.8 ± 0.2 Hz obtained from equation (11), using the experimental values for the various quantities entering into that equation.

In figure 6(b) the velocity of the cylinder versus time, $v(t)$, is also now plotted for another case: an initial up-thrust exceeding the maximum velocity allowed from the 'no-return' condition was given to the cylinder. After reaching the maximum height, the cylinder rolls down, surpassing each of the washboard potential energy minima until it eventually falls from the bottom edge of the incline.

Let us finally remark that the magnet used in our experiment could be replaced by a rectangular electric current loop around the cylinder, with its plane containing the cylinder axis [10]. However, in practice, the magnet turns out to be simpler to use.

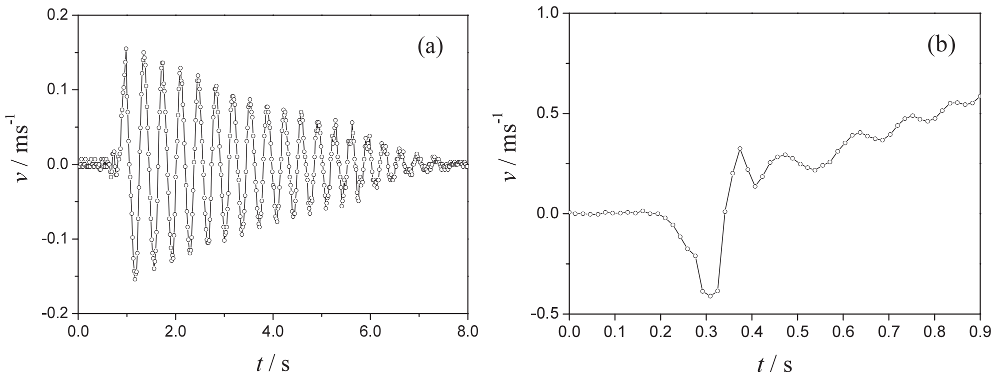

Figure 6. Velocity of the cylinder as a function of time in real experiments for $A = 2.7 \pm 0.2$ and two different initial conditions. (a) Cylinder slightly displaced from its equilibrium position and released from rest. (b) An initial up-thrust was given to the cylinder exceeding the critical velocity of 'no-return'. Note that v is positive when the cylinder moves down the plane. The data points were collected using the sonar sensor (on the left side of the picture in figure 5).

5. Conclusions

We investigated the dynamics of a cylinder placed on an inclined plane with a torque along its axis, besides the usual forces. We showed that the CM motion is governed by a tilted washboard potential, which exhibits a 'fold catastrophe'-like behaviour. Curiously, tilted washboard potentials have been recently used to describe, in simplified models, a few interesting phenomena such as Josephson tunnelling in superconducting junctions [6], quantum transport of ultra-cold atoms by laser beams [11] and diffusion in semiconductors [12].

Regarding the statics of our system, for an adequate choice of the control parameter, the cylinder remains in equilibrium. However, if the parameter decreases below the critical value, the stable equilibrium point disappears and it starts rolling down.

When the control parameter allows for a potential with equilibrium points, the system may perform oscillations around an equilibrium position, if its energy is small enough. However, for other initial kinematical conditions, endowing the system with a sufficiently high total energy, the cylinder may reach a no-return point, thereafter rolling down and moving away from the equilibrium region. This is the typical behaviour of a particle in a fold catastrophe potential.

We constructed a simple apparatus suitable for demonstrations, using a cylinder with a small magnet inside, placed on an inclined plane in a region of a homogeneous magnetic field provided by Helmholtz coils. The experimental data concerning the critical value of the control parameter and the frequency of oscillations around the potential minima are in good agreement with theory.

To the extent of our knowledge there are not many reported examples of simple one-dimensional mechanical systems showing the 'fold catastrophe' behaviour. The system presented in this paper is a good practical example of that behaviour, with the attraction that both the experiments and the computer simulations are feasible and easy to implement.

Acknowledgment

We thank Carlos Fiolhais for a critical reading of the manuscript.

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